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Compound Interest Doubling Time Rule: Extensions and Examples from Antiquities

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Abstract

Compound interest calculations are used in most financial transactions concerning loans and investments. Of special interest, is calculating the time it takes a principal to double at a certain compound interest rate. This article starts by discussing the famous Rule of 70 (or 72) that gives a simple estimate of the doubling time under compound interest. The Rule of 70 is then extended to estimate the time for a principal to grow to a higher fold (triple, quadruple, etc.) under compounding. Finally, this article shows that doubling time (or to higher folds) calculations were carried out in antiquity as evidenced by many excavated ancient cuneiform texts from Mesopotamia (c. 2000 BCE).

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1 Introduction

Interest bearing accounts can grow by earning interest according to the simple interest method or to the compound interest method. In the simple interest method, the interest is paid on the initial investment (the principal) only, not on the interest earned over time. In the compound interest method, the interest is paid on the principal and on the interest accrued over time. Thus, compound interest is called “interest on the interest.” While compound interest is used in most banking and financial transactions such as in saving accounts and loan repayments, simple interest is used in some short-term financial transactions.

Although compound interest is currently a driving force in most modern economies, it is by no means a modern invention. Cuneiform text tablets from the Sumerian period (ca. 2600-2350 BCE) and from the Old Babylonian period (ca. 2000-1600 BCE) showed concrete evidence of compound interest calculations for transactions between individuals and between city states. Some of those ancient examples demonstrate calculations of the time (doubling time) it takes a principal to double (or grow to higher folds) under compounded interest.

The doubling time of a principal is still of interest in our modern days. The present-day rule of 70 states that it will take $\frac{70}{100r}$ years for a principal to double at an annual compound interest rate of r (as a decimal). Thus at 5% annual interest rate, a principal doubles in $\frac{70}{5} = 14$ years.

In this article, we discuss the exact and approximate doubling time rules under compound interest. Then we derive exact and approximate mathematical extensions for calculating the times for a principal to triple, quadruple, quintuple, etc., under compound interest. Two interesting examples from antiquities are presented to demonstrate the early history of compound interest calculations.

2 Preliminary Notes

For the benefit of the reader, we give the basic terminology and formulae regarding present-day mathematics of compound interest:

P = the initial principal (or present value) that is invested,

r = annual interest rate expressed in decimal form. In percent form, 100r% is called the annual percentage rate (APR).

n = frequency of compounding periods per year. For example, n = 1, 2, 4, 12, 52, and 365 corresponds to annual, semi-annual, quarterly, monthly, weekly, and daily compounding, respectively. Also n = 1/2, 1/3, 1/4, 1/5, and 1/10 corresponds to compounding every two years (biannual), every three years (triennial), every four years (quadrennial), every five years (quinquennial), and every 10 years (decennial). An ancient history example of compounding once every five years is presented in this paper.

t = time of investment in years,

F = future value of the principal P, and

I = total accrued interest.

The basic compound interest formulae are

$$F = P\left(1+\frac{r}{n}\right)^{nt} \tag{1}$$

$$I = F - P, \tag{2}$$

For continuous compounding (i.e., when $n \to \infty$), the basic formula becomes

$$F = P\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t} = P[e^{r}]^{t} = Pe^{rt}. \tag{3}$$

The simple interest method has two main formulae: $F = P(1+rt)$ and $= Prt$.

3 The m-Fold Exact Time Rule

The question at hand is "How long does it take a principal to grow m-folds at an annual interest rate of *r* and a compounding frequency of *n* times per year?"

Solution: Substitute *F* = *mP* in formula (1) to obtain

$$mP = P\left(1+\frac{r}{n}\right)^{nt}$$

$$m = \left(1+\frac{r}{n}\right)^{nt}$$

$$\ln m = nt\ln\left(1+\frac{r}{n}\right)$$

$$nt = \frac{\ln m}{\ln\left(1+\frac{r}{n}\right)} \quad \text{in compounding periods.}$$

In number of years, the m-fold exact time compounding formula is

$$t = \frac{\ln m}{\text{n}\ln\left(1+\frac{r}{n}\right)} \quad \text{in years.} \tag{4}$$

For continuous compounding, the m-fold exact time formula can be derived from (3) as follows:

$$F = Pe^{rt}$$

$$mP = Pe^{rt}$$

$$m = e^{rt}$$

$$\ln m = rt\ln e$$

$$\ln m = rt$$

$$t = \frac{\ln m}{r} \quad \text{in years.} \tag{5}$$

Note that the value of m can be equal to 1.5 if one requires the time for a principal to grow by 50%. Of special interest is when m = 2, which leads to the doubling exact time formula

$$t = \frac{\ln 2}{n\ln\left(1+\frac{r}{n}\right)} \quad \text{in years} \tag{6}$$

For continuous compounding, the doubling exact time formula becomes

$$t = \frac{\ln 2}{r} \quad \text{in years.} \tag{7}$$

4 The m-Fold Approximate Time Rules

Instead of using the doubling exact time in formula (6) that requires logarithmic calculations, present-day financiers use a simple rule (the rule of 70) to approximate the required doubling time under compound interest. The rule of 70 states that at an annual interest rate of r (in decimal form), a principal doubles in $\frac{70}{100r}$ years. Thus, at a 7% APR, money doubles in $\frac{70}{7} = 10$ years. Sometimes people prefer using 7*2* instead of 70 because *72* has more divisors than does 70. The 72 double-your-money rule becomes $\frac{72}{100r}$. We will extend the rule of 70 (or 72) to rules that give the approximate time for a principal to grow m-folds (triple, quadruple, etc.).

Suppose that a principal P is invested at an annual interest rate of r (in decimal) compounded at a frequency of n > 0. It is required to determine the approximate time for the principal to grow m-folds, i.e., becomes mP, where $m \geq 1$.

Recall the m-fold exact time formula in (4), $= \frac{\ln m}{n\ln\left(1+\frac{r}{n}\right)}$.

Using the Maclaurin's series expansion (see any calculus book), we get

$$\ln\left(1+\frac{r}{n}\right) = (r/n) - \frac{(r/n)^2}{2} + \frac{(r/n)^3}{3} - \cdots \quad \text{, for } \left|(r/n)\right| < 1.$$

Because (r/n) is small, we can ignore terms involving $(r/n)^2, (r/n)^3, etc.$ Thus,

$$\ln\left(1+\frac{r}{n}\right) \cong (r/n).$$

Therefore, formula (4) becomes

$$t = \frac{\ln m}{n\ln\left(1+\frac{r}{n}\right)} \cong \frac{\ln m}{n(r/n)}.$$

The m-fold approximate time formula becomes

$$t \cong \frac{\ln m}{r}, \text{ in years.} \tag{8}$$

When m=2, we get the famous approximate double time formula (the Rule of 70) as follows:

$$t \cong \frac{\ln 2}{r} = \frac{0.6931}{r}, \text{ or } t \cong \frac{0.70}{r}, \text{ which leads to}$$

$$t = \frac{70}{100r} \text{ in years.} \tag{9}$$

Note 1: The m-fold approximate time in formulae (8) and (9) are indifferent to the compounding frequency, n.

Note 2: The m-fold t approximate and the exact time formulae for continuous compounding are identical: $t = \frac{\ln m}{r}$ in years.

Table 1 shows extensions of the Rule of 70 to m-fold approximate time rules.

Table 1: The m-fold approximate compound interest time rules in years, where r is the annual interest rate.

m	$\frac{\ln m}{r}$	m-fold approximate time rules in years	r = 0.02 years	r = 0.05 years	r = 0.10 years	r = 0.20 years
1.5	0.40547/r	40/100r rule of 40	20	8	4	2
2	0.69314/r	70/100r rule of 70	35	14	7	3.5
2.5	0.91629/r	92/100r rule of 92	46	18.4	9.2	4.6
3	1.09861/r	110/100r rule of 110	55	22	11	5.5
4	1.38629/r	140/100r rule of 140	70	28	14	7
5	1.60944/r	160/100r rule of 160	80	32	16	8
6	1.79176/r	180/100r rule of 180	90	36	18	9
7	1.94591/r	195/100r rule of 195	97.5	39	19.5	9.75

8	2.07944/r	208/100r rule of 208	104	41 .6	20.8	10.4
9	2.19723/r	220/100r rule of 220	110	44	22	11
10	2.30259/r	230/100r rule of 230	115	46	23	11.5

5 Compound Interest Time Calculations in Antiquities

Mesopotamia (Ancient Iraq in southwest Asia) is the land where the Sumerian, Akkadian, Babylonian, and Assyrian ancient civilizations had flourished and made significant imprints in the human history, especially in mathematics. Excavated cuneiform writings revealed that Mesopotamians handled problems involving arithmetic and geometric progressions, square and cube roots of integers, quadratic and simultaneous equations, Pythagoras theorem applications, areas and volumes of geometrical figures, and compound interest calculations. References on Mesopotamian (or Babylonian) mathematics include Baqir [1, written in Arabic], Neugebauer and Sachs [2], Nuegebauer [3], and Robson [4]. Boyer [5] is a general book on the history of mathematics. Roux [6] gives a short history of Mesopotamia.

In this article, we discuss two Mesopotamian artifacts that involve compound interest transactions; specifically doubling time calculations. Mesopotamians used exponential tables and their inverses (logarithmic tables) to handle *compound interest calculations*. Annual interest rates of 10% up to 33% on loans were common in Mesopotamia in early periods. Usually, archeologists assign alphanumeric identifications to the cuneiform artifacts they excavate. For example, AO stands for “Antiquite` Orientales” in the Louver in Paris and VAT stands for “Vorderasiatische Abteilung, Tontafeln” in Staatliche Museen in Berlin.

In Section 5.1, we discuss cuneiform tablet AO 6770 that calculates the time for a principal to double at 20% APR compounded annually. In Section 5.2, we discuss cuneiform tablet VAT 8528 that calculates the time for a principal to grow

64 folds when the frequency of compounding occurs once every five years (quinquennial).

5.1 Tablet AO 6770 (c. 2000 BCE, in the Louvre in Paris)

In effect, Tablet AO 6770 asks the question: *"How long does it take a principal to double at 20% APR compounded annually?"*
We will discuss the present-day, the Rule of 70, and the Mesopotamian solutions to the above question, in that order.

The Present-day Exact Solution:

Substitute *r = 20% = 0.20* and *n = 1* in the exact doubling time formula in (6) to obtain

$$\mathrm{t} = \frac{\ln 2}{n \ln\left(1+\frac{r}{n}\right)} = \frac{\ln 2}{\ln\left(1+\frac{0.20}{1}\right)} = \frac{\ln 2}{\ln(1.20)} = \frac{0.6931}{0.1823} = 3.80 \text{ years.}$$

Thus, the present-day exact answer is 3.80 years, which is equivalent to 3 years, 9 months and 18 days.

The Rule of 70 Approximate Solution:

Substitute r = 0.20 in the rule of 70, formula (9), to obtain

$$t = \frac{70}{100r} = \frac{70}{20} = 3.5\ years.$$

Thus, the rule of 70 answer is: 3 years and 6 months.

The Mesopotamian Solution:

The Mesopotamian solution in Tablet AO 6770 reduced the question to that of finding the time *t* in the exponential equation $2 = \left(\frac{6}{5}\right)^t$. This is equivalent to our present-day formula $F = P\left(1+\frac{r}{n}\right)^{nt}$, in which $= 2P, n = 1,\ and\ r = 0.20 = \frac{1}{5}$. As evidenced by excavations, the Mesopotamians had calculated tables of powers of various numbers. By searching through the power tables of $\left(\frac{6}{5}\right)$, they found that $2 = \left(\frac{6}{5}\right)^t$ leads to values of t falling between 3 and 4. Through interpolation between t = 3 and t = 4, their answer was 3 years and $9\frac{4}{9}$ months. This Mesopotamian answer is equivalent to 3 years, 9 months, 13 days and one-third of a day.

It can be seen that the Mesopotamian answer of "3 years, 9 months, 13 days and one-third of a day" is very close the present-day exact answer of "3 years, 9 months and 18 days." For further reading on tablet AO 6770, see Curtis [7] and Muroi [8].

5.2 VAT 8528 (c. 2000 BCE, in Berlin)

Tablet VAT 8528 deals with finding the time required for a principal invested at 20% APR to grow 64 folds when compounding occurs once every five years. It asks the question:

"If you lend one mina of silver at an annual interest rate of 12/60 of a mina per year, how long does it take to be repaid as 64 minas?"

In VAT 8528, the Mesopotamian compounded interest by capitalizing the interest only when the outstanding principal doubled. In VAT 8528 the annual interest rate is given as $\frac{12}{60} = 20\%$. The basic simple interest scenario formula is $F = P(1 + rt)$. Substitute $F = 2P, r = 0.20$ to obtain $2 = (1 + 0.20t)$. Solving, we see that doubling the standing principal occurs every $t = 5\ years$.

Present-day Exact Solution:

Accounting for the fact that compounding occurs once every five years, the question in VAT 8528 becomes: "*How long will it take a principal to grow 64-folds if compounding occurs once every five years at a 20% APR?"*

Substituting m = 64, r = 0.20, n = 1/5 in formula (4), we obtain

$$t = \frac{\ln m}{\mathrm{n}\ln\left(1+\frac{r}{n}\right)} = \frac{\ln 64}{\frac{1}{5}\ln\left(1+\frac{0.20}{1/5}\right)} = \frac{5\ln 64}{\ln 2} = \frac{5(4.1588)}{0.6931} = 30\ years\ .$$

The m-Fold Approximate Time Rule:

Substitute m = 64 and r = 0.20 in formula (4), the m-fold approximate time rule to obtain

$$t \cong \frac{\ln m}{r} = \frac{\ln 64}{0.20} = \frac{4.1588}{0.20} = 20.8\, years.$$

This approximation of 20.8 years is very far off the exact answer of 30 years. The reason for this gross discrepancy is that the approximation $\ln\left(1+\frac{r}{n}\right) \cong \frac{r}{n}$ requires $\frac{r}{n} < 1$. In our case, $\frac{r}{n} = \frac{0.20}{1/5} = 1.0$ and $\ln\left(1+\frac{0.20}{1/5}\right) = \ln 2 = 0.6931$ are quite different. The time approximation rules may become invalid when the compounding frequency is less than once year.

The Mesopotamian Solution:

The Mesopotamian solution in Tablet VAT 8528 involves reducing the question to that of solving the exponential equation $2^{\frac{t}{5}} = 64$. One can arrive at this equation by substituting

$F = 64, P = 1, r = 0.2, and\ n = \frac{1}{5}$ in formula (1) to obtain:

$$F = P\left(1+\frac{r}{n}\right)^{nt}$$

$$64 = \left(1+\frac{0.2}{1/5}\right)^{\frac{t}{5}}$$

$$64 = (2)^{\frac{t}{5}} \quad .$$

Then the Mesopotamians searched their power tables to look up the power of 2 such that

$2^{\frac{t}{5}} = 64.$ Knowing that $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, and $2^6 = 64$, leads to $\frac{t}{5} = 6.$ Thus the solution, t = 30 years. Again, the Mesopotamian solution of 30 years coincides with the present-day exact time solution. For further details on VAT 8528 (and a similar tablet VAT 8521), see Neugebauer [9] and Muroi [10].

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